Lyndon arrays and Lyndon trees

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Outline

1. Lyndon Arrays
2. Runs
3. Lyndon Trees
4. Cartesian Trees

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Lyndon words

Lyndon word

A non-empty word x is a Lyndon word if

x is smaller than all its proper non-empty suffixes

Lyndon arrays (1)

Lyndon array of a (non-empty) word x

For a position i on x, $i = 0, \ldots, |x| - 1$,
$Lyn[i]$ is the length of the longest Lyndon factor of x starting at i:

$$Lyn[i] = \max\{\ell \mid x[i \mathinner{..} i + \ell - 1] \text{ is a Lyndon word}\}$$

Lyndon arrays (2)

Example

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$x[i]$	a	b	b	a	b	a	b	a	a	b	a	b	b	a	b	a
$Lyn[i]$	3	1	1	2	1	2	1	8	5	1	3	1	1	2	1	1

Lyndon arrays (3)

LongestLyndon(x non-empty word of length n)

for $i \leftarrow n-1$ **downto** 0 **do**
  $(Lyn[i], j) \leftarrow (1, i+1)$
  while $j < n$ **and** $x[i \mathinner{..} j-1] < x[j \mathinner{..} j + Lyn[j] - 1]$ **do**
    $(Lyn[i], j) \leftarrow (Lyn[i] + Lyn[j], j + Lyn[j])$
return Lyn

Lyndon arrays (4)

Well known properties

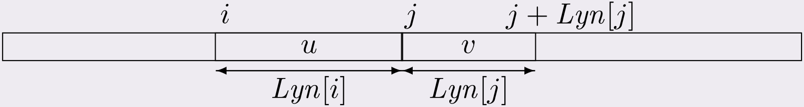
1. If u and v are Lyndon words and $u < v$ then
   - uv is also a Lyndon word
   - $u < uv < v$.
2. Each non-empty word x factorises uniquely as $u_0 u_1 u_2 \cdots$, where
   - each u_i is a Lyndon word
   - $u_0 \geq u_1 \geq u_2 \geq \cdots$.
   - u_0 is the longest Lyndon prefix of x.

Lyndon arrays (5)

Invariant

When computing $Lyn[i]$:

- $Lyn[k]$ has already been computed for $i < k < n$
- $u[i+1\,..\,j-1] \cdot u[j\,..\,j+Lyn[j]-1] \cdots$ is the Lyndon factorisation of $x[i+1\,..\,n-1]$ where $j = i+1+Lyn[i+1]$,



If $u < v$ then $u \leftarrow uv$ and $v \leftarrow$ successor of v

Outline

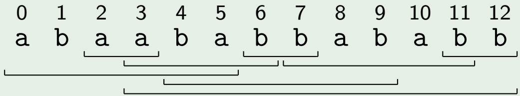
Runs (1)

Definition

A run is a maximal periodicity occurring in a word x:
interval $[i\,..\,j]$ such that

- $x[i\,..\,j]$ is periodic (i.e., its smallest period p satisfies $2p \leq |x[i\,..\,j]| = (j-i+1)$)
- the periodicity does not extend to the right nor to the left (i.e., $x[i-1\,..\,j]$ and $x[i\,..\,j+1]$ have larger periods when defined).

Runs (2)



Runs (3)

Maximal number

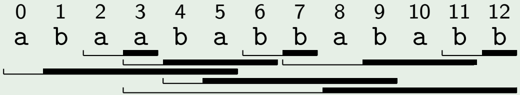
- $O(n)$ Kolpakov and Kucherov, 1999
- $5n$ Rytter, 2006
- $3.48n$ Puglisi, Simpson and Smyth, 2008
- $1.6n$ Crochemore and Ilie, 2008
- $1.49n$ Giraud, 2008
- $1.029n$ Crochemore, Ilie and Tinta, 2008
- $n - 3$ Bannai, I, Inenaga, Nakashima, Takeda, and Tsuruta, 2014
- $0.957n$ Fischer, Holub, I and Lewenstein, 2015

Runs (4)

Special position

Let us consider the orderings $<$ and its reverse $<^{-1}$.
Each run $[i \mathinner{.\,.} j]$ is associated with its greatest suffix according to one of the 2 orderings as follows:
Let $p = per(x[i \mathinner{.\,.} j])$.
If $j+1 < n$ and $x[j+1] > x[j-p+1]$ we assign to the run the position k for which $x[k \mathinner{.\,.} j]$ is the greatest proper suffix of $x[i \mathinner{.\,.} j]$ according to $<$.
Otherwise, k is the starting position of the greatest proper suffix of $x[i \mathinner{.\,.} j]$ according to $<^{-1}$.
The position k assigned this way to a run is called its special position.

Runs (5)



Runs (6)

Lemma

If the special position k of a run of period p is defined according to $<^{-1}$ (resp. $<$) then
$x[k \mathinner{.\,.} k+p-1]$ is the longest Lyndon factor of x starting at position k according to $<$ (resp. $<^{-1}$).

Runs (7)

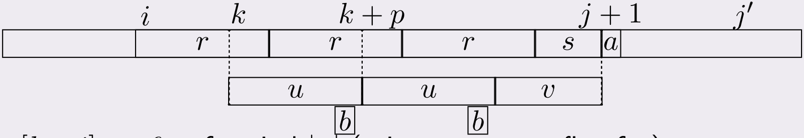
Proof

Let $[i\,..\,j]$ be a run of period p with special position k.
$x[k\,..\,k+p-1]$ is a Lyndon word because it is smaller than all its proper suffixes according to $<$.
Consider a longer factor $x[k\,..\,j']$ for $k+p \leq j' \leq j$.
It has period p which is smaller than its length; equivalently it is not border free, which shows it is not a Lyndon word for any of the 2 orderings.

Runs (8)

Proof

There is nothing else to prove if $j+1 = |x|$.
Assume then that $j' > j$ and $a = x[j+1]$.



$x[k\,..\,j] = u^e v$ of period $|u|$ (v is a proper prefix of u): greatest suffix of $x[i\,..\,j]$ according to $<^{-1}$
Since $a = x[j+1] < x[j-p+1] = b$, we get $x[k+p\,..\,j+1] < x[k\,..\,j-p+1]$, which leads to $x[k+p\,..\,j'] < x[k\,..\,j']$ and shows that $x[k\,..\,j']$ is not a Lyndon word according to $<$.

Runs (9)

RUNS(x non-empty word of length n)

for $i \leftarrow n-1$ **downto** 0 **do**
    $(Lyn[i], j) \leftarrow (1, i+1)$
    while $j < n$ and $x[i \mathinner{..} j-1] < x[j \mathinner{..} j + Lyn[j] - 1]$ **do**
        $(Lyn[i], j) \leftarrow (Lyn[i] + Lyn[j], j + Lyn[j])$
    $\ell \leftarrow |lcs(x[0 \mathinner{..} i-1], x[0 \mathinner{..} i + Lyn[i] - 1])|$
    $r \leftarrow |lcp(x[i \mathinner{..} |x|-1], x[i + Lyn[i] \mathinner{..} |x|-1])|$
    if $\ell + r \geq Lyn[i]$ **then**
        output run $[i - \ell \mathinner{..} i + Lyn[i] + r - 1]$

Runs (10)

Runs

- lcs: longest common suffix
- lcp: longest common prefix
- LCE: longest common extensions, can be computed in constant time after linear preprocessing

Outline

1. Lyndon Arrays
2. Runs
3. Lyndon Trees
4. Cartesian Trees

Lyndon trees (1)

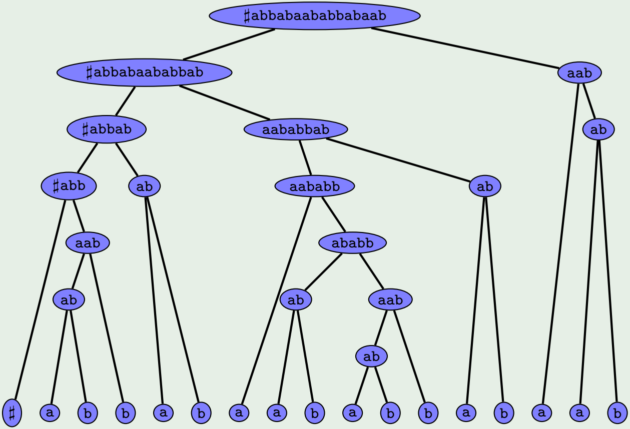
Definition

Let w be a Lyndon word s.t. $|w| > 1$, and let $w = uv$ such that v is the smallest proper non-empty suffix of w (standard factorisation) then u is also a Lyndon word.
Then the Lyndon tree $T(w)$ of w is recursively defined as follows:

- the root is w
- the left subtree of the root is $T(u)$
- the right subtree of the root is $T(v)$

Lyndon trees (2)

Example



Lyndon trees (3)

LyndonTree(x non-empty word of length n)

$(v, T(v)) \leftarrow (x[n-1], (x[n-1], (), ()))$
for $i \leftarrow n-2$ **downto** 0 **do**
    $(u, T(u)) \leftarrow (x[i], (x[i], (), ()))$
    while $u < v$ **do**
        $T(uv) \leftarrow (uv, T(u), T(v))$
        $u \leftarrow uv$
        $v \leftarrow$ next phrase or ε
return $T(x)$

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Cartesian Trees (1)

Definition

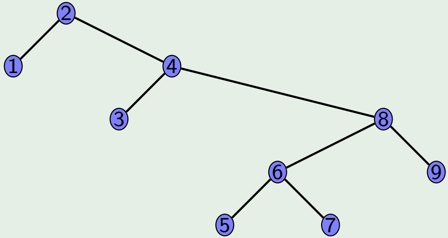
Let x be a string of numbers of length m.
The Cartesian Tree $CT(x)$ [Vuillemin 1980] of x is the binary tree where:

- the root corresponds to the index i of the minimal element of x (if there are several occurrences of the minimal element, the leftmost one is chosen)
- the left subtree is $CT(x[1 \mathinner{.\,.} i-1])$
- the right subtree is $CT(x[i+1 \mathinner{.\,.} m])$

Cartesian Trees (2)

Example

i	1	2	3	4	5	6	7	8	9
$x[i]$	3	1	6	4	8	6	7	5	9



Cartesian Trees: Incremental Construction From Right To Left, Bottom-Up Scan of the Left Path

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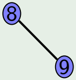

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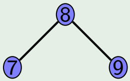

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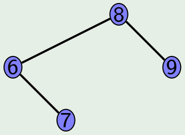

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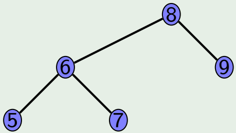

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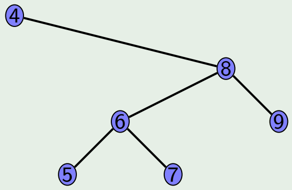

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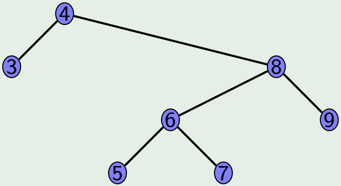

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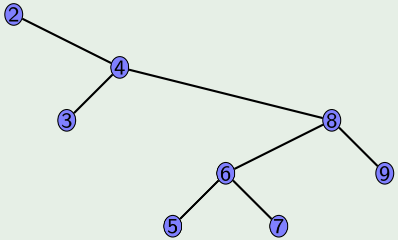

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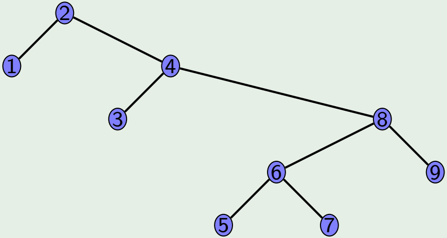


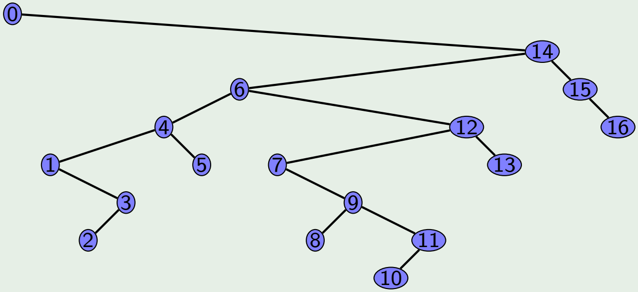
Cartesian Trees (4)

CARTESIANTREE(x non-empty word of length n)

$Root(T) \leftarrow n-1$
$S \leftarrow (n-1)$
for $i \leftarrow n-2$ **downto** 0 **do**
  $r \leftarrow Nil$
  while $S \neq \emptyset$ and $x[i] < x[Top(S)]$ **do**
    $r \leftarrow Pop(S)$
  $Right(i) \leftarrow r$
  if $S \neq \emptyset$ **then**
    $Left(Top(S)) \leftarrow i$
  else $Root(T) \leftarrow i$
return T

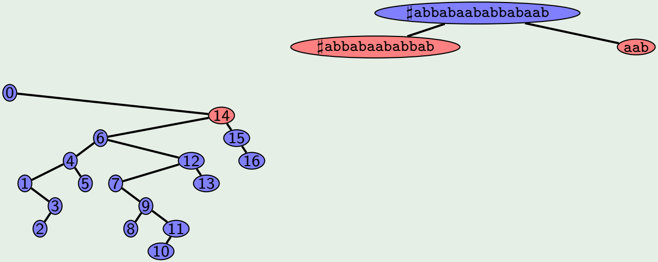
Cartesian Trees (5)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
x	$\sharp$	a	b	b	a	b	a	a	b	a	b	b	a	b	a	a	b
SA	0	14	6	15	12	4	7	9	1	16	13	5	11	3	8	10	2
ISA	0	8	16	13	5	11	2	6	14	7	15	12	4	10	1	3	9



Cartesian Trees (6)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
x	♯	a	b	b	a	b	a	a	b	a	b	b	a	b	a	a	b



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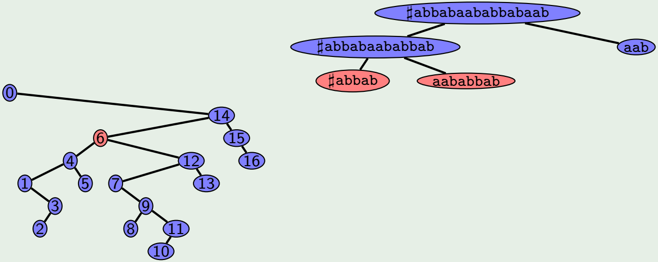
x	$\sharp$	a	b	b	a	b	a	a	b	a	b	b	a	b	a	a	b



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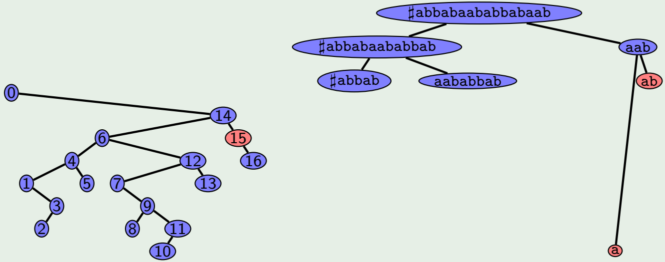

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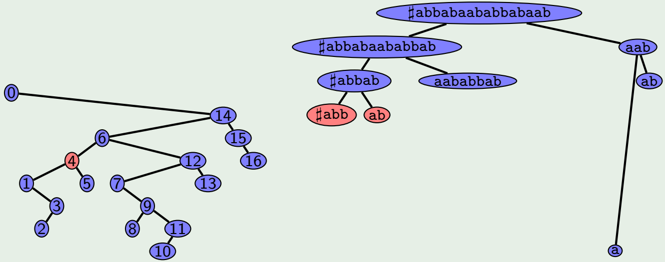

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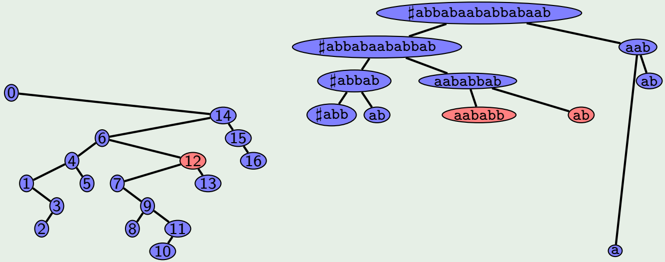

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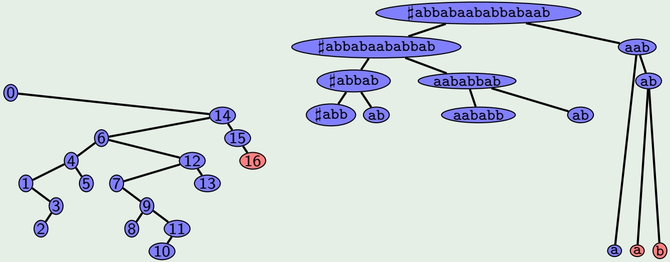

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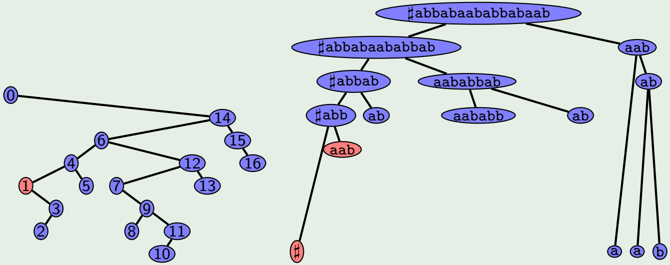
x	$\sharp$	a	b	b	a	b	a	a	b	a	b	b	a	b	a	a	b



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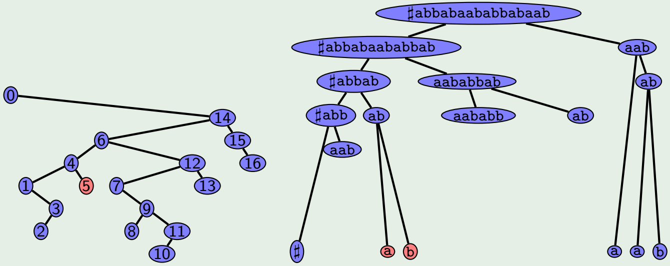

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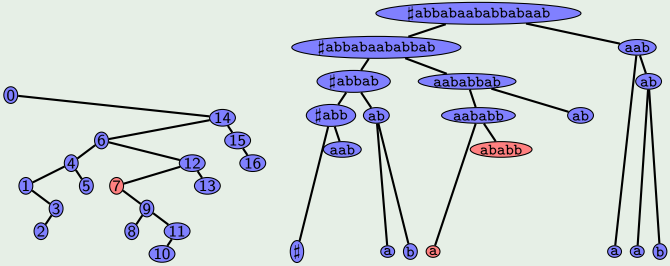

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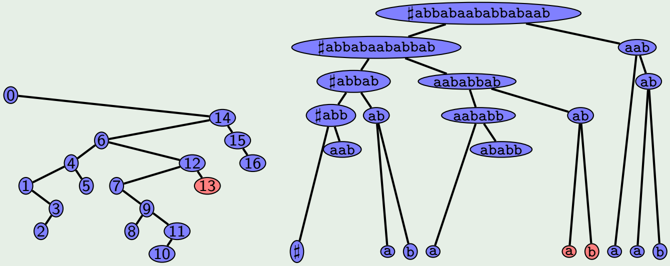

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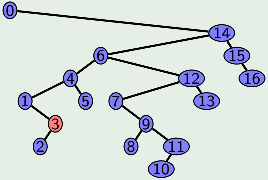

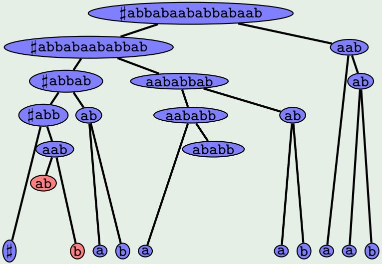

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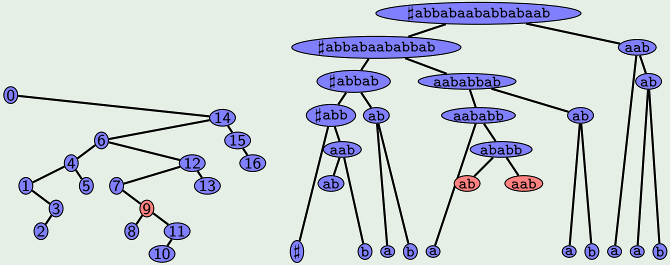

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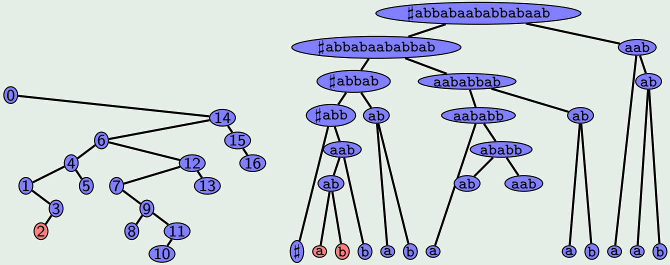

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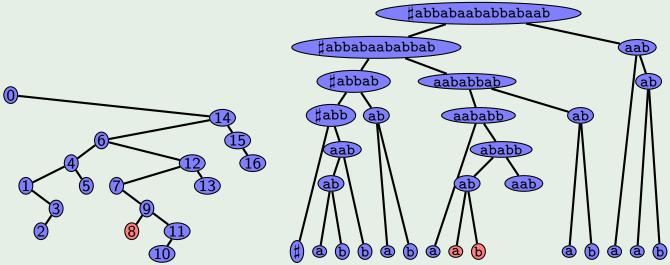

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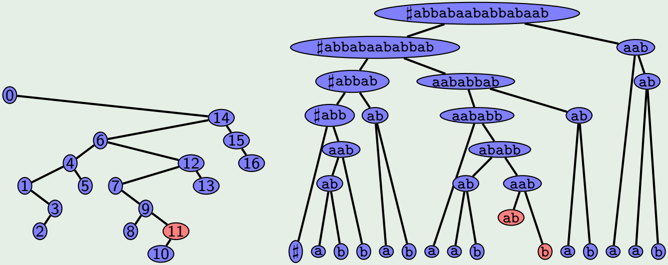

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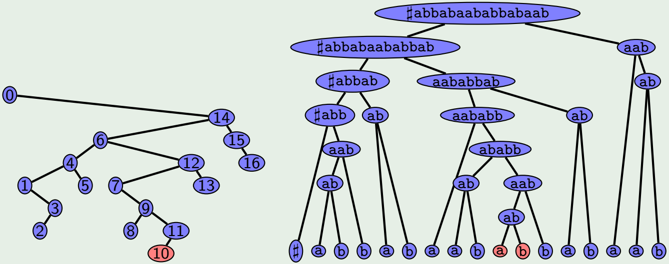
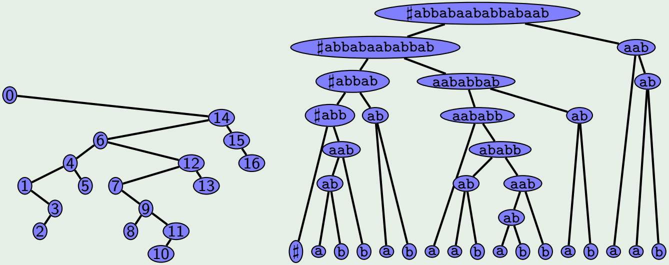
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
x	$\sharp$	a	b	b	a	b	a	a	b	a	b	b	a	b	a	a	b



Cartesian Trees (6)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
x	$\sharp$	a	b	b	a	b	a	a	b	a	b	b	a	b	a	a	b



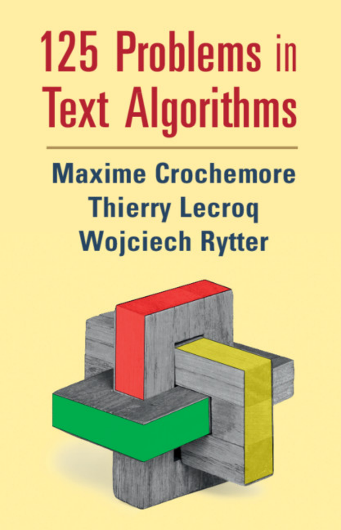
LongestLyndon(x, n)

for $i \leftarrow n-1$ **downto** 0 **do**
  $(Lyn[i], j) \leftarrow (1, i+1)$
  while $j < n$ and $x[i \,..\, j-1] < x[j \,..\, j+Lyn[j]-1]$ **do**
    $(Lyn[i], j) \leftarrow (Lyn[i]+Lyn[j], j+Lyn[j])$
return Lyn

LongestLyndon(x, n)

for $i \leftarrow n-1$ **downto** 0 **do**
  $(Lyn[i], j) \leftarrow (1, i+1)$
  while $j < n$ and $ISA[i] < ISA[j]$ **do**
    $(Lyn[i], j) \leftarrow (Lyn[i]+Lyn[j], j+Lyn[j])$
return Lyn

Main Reference



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Thank you for your attention!